Theory of ferromagnetic resonance in perpendicularly magnetized nanodisks: Excitation by the Oersted field

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We present theoretical studies of ferromagnetic resonance in perpendicularly magnetized nanodisks, wherein spin waves are excited through the ac modulation of the dc transport current injected into the disk. We have nanopillars in mind in our analysis, where spin-polarized current is injected from a metallic ferromagnet elsewhere in the structure. We argue that in a limit described, the modulation of the Oersted field generated by the transport current is the dominant spin-wave excitation mechanism, and our studies explore this limit. We calculate the critical current above which the nominal ferromagnetic state becomes unstable through studies of the linewidth of the lowest spin-wave mode, which vanishes when the critical current is reached. We find that as the applied Zeeman field H_0 is decreased from values above $4\pi M_S$, the critical current has a minimum when $H_0 \sim 4\pi M_S$ to increase for values of the external field below this value.

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I. INTRODUCTION

In recent years, there has been great interest in the study of spin dynamics in objects called nanopillars, which are nanoscale structures that typically consist of two metallic ferromagnets: one with magnetization pinned or fixed by large anisotropy and other with magnetization that is quite free to precess in response to stimuli. The two ferromagnets are separated by a conducting nonmagnetic layer. Spin-polarized current may pass from the pinned magnet into the free layer, thus exciting the spins in the free layer; the magnetization can be excited into large amplitude highly nonlinear motions.[1] It is also the case that through point-contact injection of spin-polarized current into an ultrathin film, one may excite large amplitude spin motions in the film, in the region just under the contact.[2–4] In the case of nanopillars, large amplitude motions of the magnetization produce microwave radiation. It is possible to phase lock the emission from an array of nanopillars, with the consequence that the output of the array is very much larger than that of a single nanopillar.[5–7]

It is of considerable interest to understand the nature of the spin-wave normal modes in nanopillars, since they control the response of the system in the small amplitude linear-response regime. It is also the case that the eigenvectors of the spin-wave modes may be utilized to describe aspects of the magnetization motions as one enters the nonlinear regime as well.[8] Of interest is the novel ferromagnetic resonance (FMR) experiment reported by Sankey *et al.*[9] These authors inject dc current into the free layer of a nanolayer whose strength is below the threshold to excite spontaneous oscillations of the magnetization. They then excite spin waves by superimposing a small amplitude ac current onto the dc current. For the reasons discussed in Ref. 9, spin-wave excitation by this means leaves a signature on the dc magnetoresistance, in the form of a peak when the modulation frequency sweeps through a spin-wave resonance of the free layer.

In nearly all experiments on nanopillars to date, the magnetization lies in the plane of the free layer. This is true, for example, in the ferromagnetic resonance studies reported in Ref. 9. Analyses of the nature of the spin-wave modes of in-plane-magnetized disks, along with Brillouin light-scattering studies of the modes, were reported by Gubbiotti *et al.*[10,11] We remark that the analysis of the case where the magnetization lies in plane is difficult to approach with analytic methods, so the spin dynamics is explored in these papers through the use of a micromagnetic methodology these authors developed. Included in their studies was the interesting regime wherein a vortex resides within the disk. A review article that discusses this area has appeared recently.[12] While this method provides both eigenfrequencies of the modes and eigenvectors, in fact the eigenvectors that emerge are not normalized. Because of this, it is difficult to make direct contact with experimentally measured spectra through such calculations. In the case where both exchange and dipolar interactions enter importantly into the description of the spin-wave modes, the procedure for normalizing the eigenvectors is not obvious; it should be remarked. One of us has developed a scheme by which this may be done, for a sample of arbitrary shape, for the case where the ground-state magnetization is spatially uniform.[13]

We have recently explored the nature of the ground state and also the nature of the spin-wave normal modes, for a perpendicularly magnetized disk into which a transport current is injected perpendicular to its surfaces.[14] The issue examined is the influence of the Oersted field associated with the transport current on the ground state and the spin-wave modes. We demonstrated that in the presence of the Oersted field, the ground state may be viewed as a vortex state whose physical origin and character is very different than that encountered in disks magnetized in plane. In our vortex state, the magnetization is always perpendicular to the disk surfaces right at the disk center and at its outer edges it is canted and acquires a nonzero azimuthal component parallel to the plane of the disk. As the external Zeeman field H_0 is decreased from large values to the vicinity of $4\pi M_S$ or below, the canting angle near the edge of the disk becomes large to approach $\pi/2$ for fields well below $4\pi M_S$ and the vortex core becomes concentrated at the disk center in a region whose spatial extent is roughly the exchange length. The spin-wave analysis shows the vortex state to be locally stable with respect to small perturbations down to zero applied Zeeman field.

In this paper, we present an analysis of the ferromagnetic resonance spectrum of such a perpendicularly magnetized disk, where as in Ref. 9 the spin waves are excited by imposing an ac component onto the dc transport current. We argue that in a parameter regime outlined below, it is the modulation of the Oersted field that is the dominant source of spin-wave excitation, when compared to the Slonczewski[15] spin torque term. The calculations we present apply to this regime, where our previous description of the Oersted field-induced vortex state is valid. We shall see that the modes excited have a distinctly different character than encountered in classical ferromagnetic resonance, where spin waves are excited by an external microwave field. In our study of the Oersted field-induced ferromagnetic resonance, the Green's function method we develop properly incorporates the normalization of the eigenvectors, so the relative intensity of the various modes in our calculated spectra is rendered correctly. Through the study of the linewidth of the lowest FMR mode as the dc current is increased, we may calculate the critical current of our disk as a function of the strength of the applied Zeeman field. We find that the critical current assumes a minimum value when H_0 is near $4\pi M_S$ and increases substantially as the applied field is reduced.

We now turn to our analysis. In Sec. II we discuss the formalism we have developed, Sec. III presents our numerical studies, and final comments are included in Sec. IV.

II. ANALYSIS

We consider a disk of radius R and thickness d magnetized perpendicular to its surfaces; the saturation magnetization is then $\hat{z}M_S$ in the quiescent state, when transport current is absent. An external Zeeman field H_0 is also applied parallel to the z axis. We then impose a spin-polarized transport current $I(t)=I_0+\delta I(t)$, with the current density $\vec{J}=\hat{z}[I(t)/\pi R^2]$ uniformly distributed over the disk. We assume that the disk is sufficiently thin that all magnetization components are independent of the coordinate z—the direction normal to the film surfaces—and depend only on $\vec{\rho}$—the coordinate in the plane. Our interest is in the equation of motion for the magnetization $\vec{M}(\vec{\rho},t)$, which we write in the form

$$\frac{d\vec{M}(\vec{\rho},t)}{dt} = \gamma[\vec{H}(\vec{\rho},t) \times \vec{M}(\vec{\rho},t)] + \alpha\frac{\vec{M}(\vec{\rho},t)}{M_S} \times \frac{\partial\vec{M}(\vec{\rho},t)}{\partial t} + \gamma\vec{H}_{ST}(\vec{\rho},t) \times \vec{M}(\vec{\rho},t). \tag{1}$$

In the first term, $\vec{H}(\vec{\rho},t)=\vec{H}^{(T)}(\rho,t)+\vec{h}_d(\vec{\rho},t)+\frac{D}{M_S}\nabla^2\vec{M}(\vec{\rho},t)$. The third term in $\vec{H}(\vec{\rho},t)$ is the exchange effective field, with D as the exchange stiffness, $\vec{h}_d(\vec{\rho},t)$ is the dipolar field set up by the time-dependent motion of the magnetization, and $\vec{H}^{(T)}(\vec{\rho},t)$ is the vector sum of the applied Zeeman field $\hat{z}H_0$ and the Oersted field generated by the transport current which has the form $\hat{\varphi}[2I(t)\rho/cR^2]$, and then there is the demagnetizing field we write here as $-4\pi M_z(\rho)\hat{z}$. In our previous paper,[14] we discussed the influence of corrections to this local approximation to the demagnetizing field on the spin-wave spectrum of the disk. These corrections referred to as gradient corrections led to rather small quantitative corrections to the spin-wave frequencies. We thus set the gradient corrections aside in the present paper, in the interest of simplicity. The second term on the right-hand side of Eq. (1) is the Gilbert form of the phenomenological damping term and the third term is the spin torque term, with the origin in the fact that the transport current injected into the film is spin polarized.[15,16] We follow Rezende *et al.*[17] by writing the spin torque effective field as $\vec{H}_{ST}(\vec{\rho},t)=[\varepsilon\hbar I(t)/2\pi M_S^2R^2de]\times[\vec{M}(\vec{\rho},t)\times\hat{n}]$. Here ε is the degree of spin polarization in the transport current injected into the "active" disk, e is the magnitude of the electron charge, and the vector $\hat{n}$ is in the direction of the injected spin current. Again we follow the authors of Ref. 17 by choosing $\varepsilon=0.2$ in our numerical estimates and studies, and we write $\hat{n}=\cos\theta_0\hat{x}+\sin\theta_0\hat{z}$. The spin polarization of the transport current will in general be out of the plane of the disk by virtue of the Zeeman field that is perpendicular to the film surfaces. Our final results do not depend sensitively on the value of θ_0, however.

We proceed, as in Ref. 14, to linearize Eq. (1) about the ground state in the presence of the transport current. One proceeds by writing the magnetization components $M_\alpha(\vec{\rho},t)=M_\alpha^{(E)}(\vec{\rho})+m_\alpha(\vec{\rho},t)$. Here $M_\alpha^{(E)}(\vec{\rho})$ is the equilibrium magnetization, which is tilted away from the z direction by the Oersted field associated with the injected current, and as we shall discuss shortly by the spin torque term as well. Then $m_\alpha(\vec{\rho},t)$ is the amplitude of the spin wave, which we assume to be small. Thus one proceeds by linearizing Eq. (1) with respect to the small amplitude spin-wave motion described by $m_\alpha(\vec{\rho},t)$. As one proceeds with this process, one encounters terms zero order in $m_\alpha(\vec{\rho},t)$. These are set to zero, and when this is done one has in hand a set of differential equations that determine the form of the ground-state magnetization as described by $M_\alpha^{(E)}(\vec{\rho})$. There is an issue that must be discussed before we turn to a summary of the details of this procedure.

In our previous publication,[14] we explored the influence of the Oersted field on the ground state of the perpendicularly magnetized disk. This led us to the vortex state whose character was discussed in Sec. I. The magnetization in the ground state is here invariant under rotation about the z axis. That is the static magnetization in the ground state $\vec{M}^{(E)}(\vec{\rho})$ depends only on the distance from the origin $\rho=|\vec{\rho}|$. In fact $\vec{M}^{(E)}$ has only a $\hat{z}$ component and a $\hat{\varphi}$ component, so we wrote the ground-state magnetization in the form $\vec{M}^{(E)}(\rho)=M_S[\hat{\varphi}\sin\psi(\rho)+\hat{z}\cos\psi(\rho)]$. A differential equation that may be solved for $\psi(\rho)$ was derived in Ref. 14. We then explored spin-wave excitations out of the vortex ground state through the use of the linearized Landau-Lifschitz equation, with both Gilbert and spin torque "antidamping" set aside. In this framework, the spin-wave normal modes are characterized by an azimuthal quantum number m, and in the eigenvectors one encounters the factor $\exp(im\varphi)$ found in analyses of geometries with rotational symmetry about z.

The picture in the previous paragraph is rendered more complex when the spin torque term is included in the equation of motion. When one linearizes the equation of motion, the zero-order equation contains a contribution proportional to $\vec{M}^{(E)}(\vec{\rho})\times[\hat{n}\times\vec{M}^{(E)}(\vec{\rho})]$. This breaks the cylindrical symmetry of the problem. If the spin torque magnetic field $\vec{H}_{ST}(\vec{\rho},t)$ is comparable in magnitude to the Oersted field, the ground state will be complex in nature and thus influenced importantly by the direction of the spin polarization of the injected current. Under such circumstances, one can make few general statements regarding either the ground state or the nature of the spin dynamics in nanodisks such as those considered in this paper.

For this reason, the analysis and numerical calculations reported here confine their attention to the circumstance where the effective spin torque field is modest in magnitude compared to the Oersted field. When the equations of motion are reduced to dimensionless form, the ratio of the spin torque effective field to the Oersted field is controlled by the dimensionless parameter $\eta=\varepsilon\hbar c/4\pi M_S Rde$. We thus confine our attention to the case where this parameter is small compared to unity. The spin torque "antidamping" term will be comparable to the Gilbert damping term, but our interest resides in the case where the terms in the equation of motion contributed by the Oersted field are considerably larger than these two.

The authors of Ref. 4 introduce a spin torque term identical to that in our Eq. (1), in their discussion of the response of a perpendicularly magnetized film to a spatially localized source of spin-polarized current injected into the film. These authors assume that at all times the magnetization has rotational symmetry about the z axis, which is located at the center of the circular disk into which the current is injected. This assumption, unfortunately, is incompatible with the symmetry of the original Landau-Lifschitz equation that is the starting point of their analysis, after the spin torque term is introduced.

In the limit that the dimensionless parameter η is small compared to unity then to good approximation the ground-state configuration of the nanodisk is well approximated by the vortex state presented in Ref. 14. The ground-state magnetization has the form $M_S[\sin\psi(\rho)\hat{x}+\cos\psi(\rho)\hat{z}]$, where the angle $\psi(\rho)$ is found by solving the equation derived in Ref. 14,

$$\frac{D}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)=\frac{D}{2\rho^2}\sin(2\psi)+H^{(I)}(\rho)\sin\psi(\rho) - H^{(\mathrm{Oe})}(\rho)\cos\psi(\rho), \tag{2}$$

where $H^{(I)}(\rho)=H_0-4\pi M_S\cos\psi(\rho)$ and the Oersted field is $H^{(\mathrm{Oe})}(\rho)=2I\rho/cR^2$. The boundary conditions are $\psi(0)=0$ and for reasons discussed in Ref. 14 $\partial\psi/\partial\rho|_R=0$.

We may then proceed with the linearization process. We proceed very much as in Ref. 14, with the addition of the Gilbert damping term, and the spin torque term. We erect a local coordinate system at each point in the disk, with one axis parallel to the local magnetization that lies in the plane formed by the unit vectors $\hat{z}$ and $\hat{\varphi}$, a second axis in the radial direction $\hat{\rho}$, and the third—designated by appending the subscript t to vector components parallel to it—also lies in the plane formed by $\hat{z}$ and $\hat{\varphi}$. One finds two linearized equations in the variables m_ρ and m_t. The Gilbert damping term and the spin torque term then add two terms to the right-hand side of Eqs. (6a) and (6b) of Ref. 14 after seeking solutions where the magnetization components have the time dependence $\exp(-i\Omega t)$. We denote these terms with the symbols $\dot{m}_\rho|_{\mathrm{damp}}$ and $\dot{m}_t|_{\mathrm{damp}}$,

$$\dot{m}_\rho|_{\mathrm{damp}}=+i\alpha\Omega m_t+H_{ST}[\sin\theta_0\cos\psi(\rho) - \cos\theta_0\sin\psi(\rho)\sin\varphi]m_\rho, \tag{3a}$$

$$\dot{m}_t|_{\mathrm{damp}}=-i\alpha\Omega m_\rho+H_{ST}[\sin\theta_0\cos\psi(\rho) - \cos\theta_0\sin\psi(\rho)\sin\varphi]m_t. \tag{3b}$$

Here $H_{ST}=\varepsilon\hbar I/2\pi M_S R^2 de$. When $\eta\ll 1$ the current I in the spin torque term is replaced by its dc value.

One should note the terms in Eq. (3) that depend on the azimuthal angle φ. These are an illustration once again that the spin torque term breaks the radial symmetry in the problem, save for the very special case that the injected current has its spin polarization perpendicular to the film surfaces ($\theta_0=\pi/2$).

In what follows, we refer the reader to Eq. (6) of Ref. 14, along with the definitions of the various quantities that enter. The driving term that excites the magnetization is the oscillatory component of the Oersted field mentioned earlier, which has the form $\hat{\varphi}[2\delta I\rho/R^2c]\exp(-i\Omega t)$. Its role in driving the magnetization may be incorporated into Eq. (6) of Ref. 14 by replacing the quantity $h_\varphi^{(d)}$ in Eq. (6a) by $2\delta I\rho/R^2c$. If we then define the two quantities $H_{ST}^{(a)}(\rho)=H_{ST}\sin\theta_0\cos\psi(\rho)$ and $H_{ST}^{(b)}(\rho)=H_{ST}\cos\theta_0\sin\psi(\rho)$, then one finds

$$\left[i\Omega+H_{ST}^{(a)}(\rho)+\frac{2D}{\rho^2}\cos\psi(\rho)\frac{\partial}{\partial\varphi}\right]m_\rho-H_{ST}^{(b)}(\rho)\sin(\varphi)m_\rho + 4\pi M_S\sin^2\psi(\rho)m_t-\Bigg[H^{(T)}(\rho)-i\alpha\Omega - D\left(\nabla^2-\left\{\frac{[\cos\psi(\rho)]^2}{\rho^2}+\left[\frac{\partial\psi(\rho)}{\partial\rho}\right]^2\right\}\right)\Bigg]m_t = -\frac{2M_S\delta I}{cR^2}\rho\cos\psi(\rho) \tag{4a}$$

and

$$\left[H^{(T)}(\rho)-i\alpha\Omega-D\left(\nabla^2-\frac{1}{\rho^2}\right)\right]m_\rho+\left[i\Omega+H_{ST}^{(a)}(\rho) + \frac{2D}{\rho^2}\cos\psi(\rho)\frac{\partial}{\partial\varphi}\right]m_t-H_{ST}^{(b)}(\rho)\sin(\varphi)m_t=0. \tag{4b}$$

The next step is to write $m_{\rho,t}=\Sigma_m m_{\rho,t}^{(m)}(\rho)\exp(im\varphi)$, so we have

$$L_1^{(m)}(\rho)m_\rho^{(m)}(\rho)+\frac{i}{2}H_{ST}^{(b)}(\rho)\times[m_\rho^{(m-1)}(\rho)-m_\rho^{(m+1)}(\rho)]+L_{2a}^{(m)}(\rho)m_t^{(m)}=-\frac{2M_S\delta I}{cR^2}\rho\cos\psi(\rho)\delta_{m,0} \tag{5a}$$

and

$$L_{2b}^{(m)}(\rho)m_\rho^{(m)}(\rho)+L_1^{(m)}(\rho)m_t^{(m)}(\rho)+\frac{i}{2}H_{ST}^{(b)}(\rho)\times[m_t^{(m-1)}(\rho)-m_t^{(m+1)}(\rho)]=0. \tag{5b}$$

where we have the differential operators

$$L_1^{(m)}(\rho)=i\Omega+H_{ST}^{(a)}(\rho)+\frac{2iDm}{\rho}\cos\psi(\rho), \tag{6a}$$

$$L_{2a}^{(m)}(\rho)=-H^{(T)}(\rho)+i\alpha\Omega+D\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\{m^2+[\cos\psi(\rho)]^2\}-\left(\frac{\partial\psi(\rho)}{\partial\rho}\right)^2\right]+4\pi M_s\sin^2\psi(\rho), \tag{6b}$$

and

$$L_{2b}^{(m)}=H^{(T)}(\rho)-i\alpha\Omega-D\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{m^2+1}{\rho^2}\right]. \tag{6c}$$

Of interest to us is the $m=0$ channel, since the Oersted field excites these modes by virtue of its rotational symmetry. Thus, the modes excited in this picture are very different in nature than those excited in classical microwave ferromagnetic resonance, where the microwave field will couple to only the $m=\pm1$ modes in the limit where the microwave exciting field may be viewed as spatially uniform. Now we also see that in the presence of the spin torque terms in the equation of motion, the spin-wave eigenmodes, strictly speaking, are not characterized by a single azimuthal quantum number. Indeed, we have here an infinite hierarchy of coupled equations that describe the spin-wave modes. However, in the limit explored in this paper, where the spin torque field is viewed as small compared to the other magnetic fields in the problem, the admixture of $m=\pm1$ modes into the $m=0$ modes will be a small effect. In the limit that the amplitude of the spin torque field is small, it is straightforward to develop a perturbation theoretic description of this admixture. One writes out the equations which describe the $m=\pm1$ amplitudes, and the term which involves the $m=0$ amplitude acts as a driving term which excites the $m=\pm1$ modes. When the amplitudes of the $m=\pm1$ modes are fed back into the equation for the $m=0$ amplitude, we have terms proportional to $(H_{ST}^{(b)})^2$. In this paper, where we explore the limit where the spin torque fields are small, we may confine our attention to terms linear in the amplitude of the spin torque field. Thus, we proceed by ignoring the role of the terms in H_{ST}^b. In this limit, it remains the case that the spin-wave eigenmodes may be described to very good approximation as modes characterized by the azimuthal quantum number m and, as mentioned earlier, the time-dependent component of the Oersted field excites only the manifold of $m=0$ modes.

We then address a structure that we may write in the form

$$\begin{pmatrix} L_1^{(0)}(\rho) & L_{2a}^{(0)}(\rho) \\ L_{2b}^{(0)}(\rho) & L_1^{(0)}(\rho)\end{pmatrix}\begin{pmatrix} m_\rho^{(0)}(\rho) \\ m_t^{(0)}(\rho)\end{pmatrix}=\begin{pmatrix} f(\rho) \\ 0\end{pmatrix}, \tag{7}$$

where $f(\rho)=-\frac{2M_S\delta I}{cR^2}\rho\cos\psi(\rho)$. Such a structure may be solved by presenting a pair of Green's functions that satisfy

$$\begin{pmatrix} L_1^{(0)}(\rho) & L_{2a}^{(0)}(\rho) \\ L_{2b}^{(0)}(\rho) & L_1^{(0)}(\rho)\end{pmatrix}\begin{pmatrix} G_\rho^{(0)}(\rho,\rho') \\ G_t^{(0)}(\rho,\rho')\end{pmatrix}=\begin{pmatrix} \delta(\rho-\rho') \\ 0\end{pmatrix}. \tag{8}$$

Once the Green's functions are constructed, we have

$$m_\rho^{(0)}(\rho)=-\frac{2M_S\delta I}{cR^2}\int_0^R G_\rho^{(0)}(\rho,\rho')\rho'\cos\psi(\rho')d\rho' \tag{9a}$$

and

$$m_t^{(0)}(\rho)=-\frac{2M_S\delta I}{cR^2}\int_0^R G_t^{(0)}(\rho,\rho')\rho'\cos\psi(\rho')d\rho'. \tag{9b}$$

In Ref. 14, we argued that appropriate boundary conditions for such a nanodisk are that $m_\rho^{(0)}(R)=0$ and $\partial m_t^{(0)}/\partial\rho|_R=0$. These boundary conditions assume strong surface anisotropy at the edge of the disk, with the anisotropy axis normal to the edge of the disk. Until a theory such as that developed here can be brought into direct contact with data, the actual form of the boundary condition at the disk edge is not known; we regard this choice as reasonable from the physical point of view. We remark that our principal conclusions are not affected sensitively by the choice of boundary condition at the edge of the disk. These boundary conditions will be imposed if we require $G_\rho^{(0)}(R,\rho')=0$ and $\partial G_t^{(0)}(\rho,\rho')/\partial\rho|_{\rho=R}=0$. Analysis of the leading behavior of Eq. (8) close to the origin implies that the two Green's functions vanish at $\rho=0$ for fixed ρ'. For this class of problem, the construction of the Green's functions is not straightforward. We develop a means by which this may be done in Appendix A.

III. STUDIES OF THE FMR SPECTRUM OF A PERPENDICULARLY MAGNETIZED DISK

In this section, we present our numerical studies of the FMR spectrum of a uniformly magnetized disk, along with related issues. As noted in Sec. II, the theoretical treatment outlined above is applicable to disks for which the dimensionless parameter η is small compared to unity. In this section, we shall present numerical studies of a model Permalloy disk with a radius of 150 nm and a thickness of 10 nm. If, following the authors of Ref. 17 we take the spin transfer efficiency ε to be 0.2, then the parameter $\eta=0.16$. The exchange stiffness D has been chosen such that the exchange length $l_{\text{ex}}=(D/4\pi M_S)^{1/2}$ is equal to 6 nm, which is appropriate to Permalloy. The Zeeman field, of course, is perpendicular to the disk, and this will tip the magnetization of the pinned layer in a nanopillar out of plane. To simulate the role of the out-of-plane component of the spin torque field, we

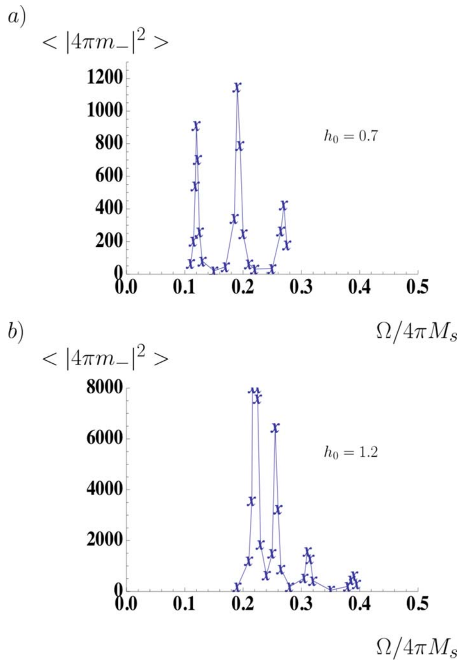

FIG. 1. (Color online) We show the ferromagnetic resonance spectrum of the model disk described in the text for two values of the Zeeman field. The quantity $h_0 = H_0/4\pi M_S$ with H_0 the Zeeman field applied perpendicular to the surface of the disk. In (a), the dc current has been taken to be 15 mA, and in (b) the dc current has been taken to be 10 mA (notice that the first peak is off scale, its value is on the order of 25 000).

have taken the angle $\theta_0 = 30°$. Strictly speaking, of course, θ_0 will vary with the applied magnetic field. To include this effect in our calculations, we need quantitative information on magnetic properties of the pinned layer. We do not have such information at present for any actual sample; it will be a straightforward matter to improve this aspect of our simulation when data on real samples are available. In our calculations, we have scaled the various magnetic fields to $4\pi M_S$. We have taken the dimensionless measure α in the Landau-Lifschitz equation to be 0.02.

In Fig. 1, we show a model FMR spectrum for the disk just described, for two values of the dimensionless Zeeman field $h_0 = H_0/4\pi M_S$. We display results for $h_0 = 1.2$ and $h_0 = 0.7$. We see three modes clearly, with a very small feature that is the fourth mode. Qualitatively the spectra bear a resemblance to those published in Ref. 9, but of course no meaningful comparison is possible since the samples used in this experiment are elliptical in shape and magnetized in plane, as opposed to the geometry analyzed in the present paper. We remark that we use as a measure of the FMR response of the disk the quantity $\langle |4\pi m_-|^2 \rangle = \{\int_0^R d\rho\rho |4\pi m_-|^2\}/R^2$, where $m_- = m_\rho - i m_t$. This is the dominant contribution to the spin-wave eigenvector, and in the absence of mixing between m_- and $m_+ = m_\rho + i m_t$ provided by the spin torque term, it is the only component of the eigenvector. Its average over the disk thus serves as a sensible measure of the amplitude of the FMR signal.

Much information on the character of the spin waves in the disk may be extracted from the Wronskian contained in the Green's function [the quantity $W(R)$ described in Appendix A]. In the absence of damping, the Wronskian has zeros on the real axis at each of the spin-wave frequencies with azimuthal quantum number $m = 0$. Of course, one may easily use the method outlined in Appendix A for calculating the Wronskian associated with any value of the azimuthal quantum number m, and the zeros of each of these provide one with the spin-wave frequencies for the appropriate azimuthal quantum number. Thus, if one wishes to generate the frequencies of the spin-wave normal modes rather than address the eigenvalue problem directly through the solution of the differential equations for the spin-wave modes as we did in Ref. 14, one may recover the same information through the use of the Wronskian. We remark that from the computational point of view, the use of the Wronskian in this matter is less demanding.

When damping is added to the problem, the zeros of the Wronskian move off the real axis of the Ω plane into the lower half plane. In our studies, in the presence of damping, we study the width of the spin-wave modes by plotting the quantity $F = 1/|W(R)|$ as a function of frequency. This function has peaks centered at the spin-wave frequencies, and the full width of the peaks at half maximum provides us with a measure of the linewidth of the mode. We proceed by fitting these peaks to a Lorentzian, very much as experimentalists do, and from this we extract the linewidth of each mode. As the dc current is turned on in our calculations and the spin torque effects assert themselves, the spin-wave linewidths narrow. The poles in the Wronskian migrate toward the real axis in the complex Ω plane as the dc current is increased. For any given spin-wave mode, at a certain critical current, the linewidth collapses to zero; the zero in the Wronskian lies on the real axis at this point. As the current is increased above the critical current, the zero of the Wronskian migrates into the upper half plane. Once this happens, the vortex state is unstable. The linearized equations of motion for m_ρ and m_t admit solutions which increase exponentially in time in this current regime. Application of the dc current will set the magnetization into oscillatory motion when the current exceeds the critical value. Of course, our theory breaks down at this point. But through the method just described we can outline the current regime in which the vortex state is stable, and we can calculate the critical current above which spontaneous oscillations set in.

In Fig. 2 below, for the two applied magnetic fields employed in Figs. 1(a) and 1(b), we show the linewidth of the first two spin-wave modes as a function of dc current for our model disk. We see that the lower of the two modes goes unstable first. Then in Fig. 3 we provide the critical current above which the vortex state is unstable as a function of applied magnetic field. What is striking is the minimum in critical current very near the applied field of $H_0 = 4\pi M_S$. In Ref. 14, we found that when appreciable dc current is present, the spin-wave frequencies display a minimum at or very near this field. As the Zeeman field is lowered below $4\pi M_S$, the modes stiffen and this suggests that the vortex state becomes more stable as the field is lowered.

We can generate eigenvectors for the spin-wave modes through the use of Eq. (9). The eigenvectors for the various modes may be calculated from these relations upon setting

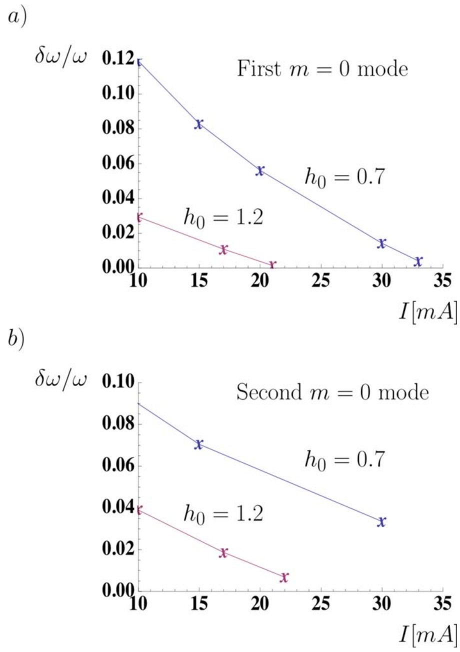

FIG. 2. (Color online) We show the linewidth of (a) the lowest $m=0$ and (b) the second $m=0$ spin-wave mode as a function of dc current, for the two applied Zeeman fields used in the calculations of the spin-wave spectrum in Fig. 1. We give the ratio of the full width at half maximum to the spin-wave frequency in these figures.

the frequency equal to that of the peaks in the FMR response displayed in Fig. 1. In Figs. 4 and 5 we illustrate the nature of the eigenvectors of the first two $m=0$ modes. As expected, the low-frequency mode is nodeless, while the second mode has a single node. We see that as the Zeeman field is lowered, the peaks in the eigenvectors are drawn in toward the center of the disk, where the vortex core resides. As the applied field is lowered, the effective field near the center of the disk becomes weaker; the central region acts as a potential well which deepens as the field is lowered. The lowest frequency mode is drawn inward as the well deepens, and the first maximum of the second mode behaves similarly. We do not show the eigenvector of the third mode. While this mode appears in our calculated spectra, its behavior is qualitatively similar to the first two low-lying modes just illustrated except—of course—it has two nodes.

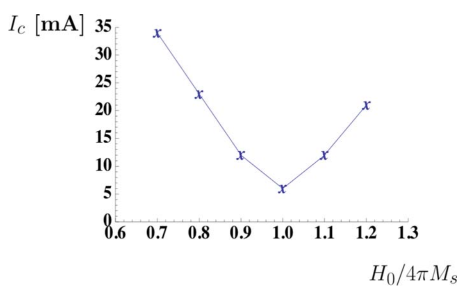

FIG. 3. (Color online) The critical current as a function of applied magnetic field, for the model disk discussed in the text. The instability is controlled by the $m=0$ spin wave, whose linewidth is the first to go to zero.

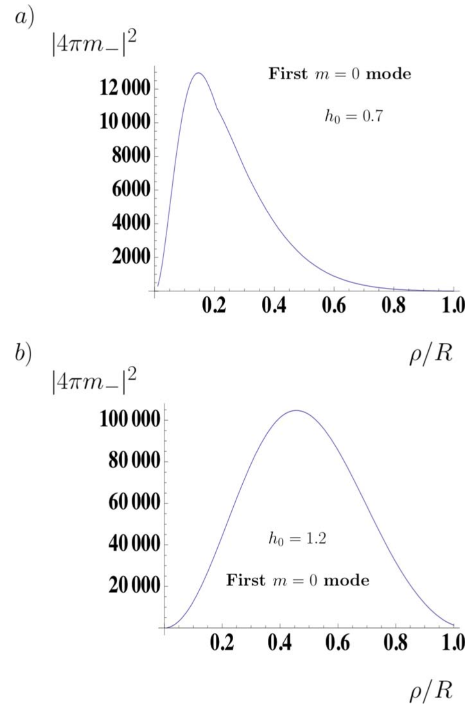

FIG. 4. (Color online) The radial variation in the dominant contribution to the eigenvector of the lowest frequency spin-wave eigenmode in the $m=0$ manifold, for the two magnetic fields used in the calculation of the FMR spectrum in Fig. 1. As in Fig. 1, the dc current assumes the value of 15 mA for the lower of the two applied fields and 10 mA for the higher applied field.

IV. RESULTS AND DISCUSSION

We have developed a theoretical structure that allows us to discuss the spin dynamics and ferromagnetic resonance spectrum in perpendicularly polarized nanodisks, wherein the excitation of the spin waves has its origin in ac modulation of the transport current injected into the disk. The explicit calculations we present focus on samples in which the modulation of the Oersted field is the dominant source of excitation. As we pointed out, under circumstances where the effective magnetic field associated with the spin torque term is comparable to the Oersted field, the presence of spin polarization in the injected current breaks the rotational symmetry of the disk; while in principle our method will apply to this situation, it will be necessary to solve numerically a hierarchy of radial equations. As we noted in Sec. II, the authors of Ref. 4 have overlooked this complication, so far as we can see. It is our view that the methodology set forth here can be readily adapted to a full discussion of the general problem. We plan to turn to this in the future.

Our method allows us to calculate the critical current by exploring the spin dynamics in the disk in the low current stable vortex state presented in Ref. 14 and then increasing the current until the linewidth of the lowest-lying spin-wave mode vanishes. We find the striking dependence of the critical current on applied magnetic field displayed in Fig. 3, where the critical current is not a monotonic function of applied magnetic field but rather has a minimum for applied fields H_0 in the near vicinity of $4\pi M_S$. We will be most

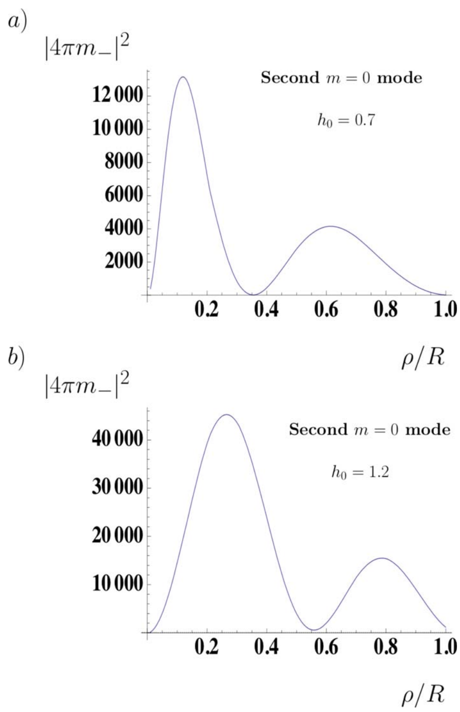

FIG. 5. (Color online) The radial variation in the dominant contribution to the eigenvector of the second spin-wave eigenmode in the $m=0$ manifold, for the two magnetic fields used in the calculation of the FMR spectrum in Fig. 1. As in Fig. 1, the dc current assumes the value of 15 mA for the lower applied field and 10 mA for the higher field.

interested to see the experimental study of samples such as those explored here.

ACKNOWLEDGMENTS

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APPENDIX A: CONSTRUCTION OF THE GREEN'S FUNCTIONS

In what follows, the two functions $G_\rho^{(0)}(\rho,\rho')$ and $G_t^{(0)}(\rho,\rho')$ will be considered to be the functions of the variable ρ with ρ' held fixed. We will make use of solutions of the homogeneous equations

$$\begin{pmatrix} L_1^{(0)}(\rho) & L_{2a}^{(0)}(\rho) \\ L_{2b}^{(0)}(\rho) & L_1^{(0)}(\rho) \end{pmatrix} \begin{pmatrix} \tilde{m}_\rho^{(0)}(\rho) \\ \tilde{m}_t^{(0)}(\rho) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A1}$$

where we have added the tilde above the m to distinguish these special functions from the physical magnetization of the sample. We may construct four independent solutions $\{\tilde{m}_{\rho i}^{(0)}(\rho), \tilde{m}_{ti}^{(0)}(\rho)\}$ through the use of "one-sided" boundary conditions as follows:

$$\tilde{m}_{\rho 1}^{(0)}(0) = \tilde{m}_{t1}^{(0)}(0) = 0, \quad \tilde{m}_{\rho 2}(0) = \tilde{m}_{t2}(0) = 0, \tag{A2a}$$

$$\left.\frac{\partial \tilde{m}_{\rho 1}^{(0)}}{\partial \rho}\right|_0 = 1, \quad \left.\frac{\partial \tilde{m}_{t1}^{(0)}}{\partial \rho}\right|_0 = 0, \quad \left.\frac{\partial \tilde{m}_{\rho 2}^{(0)}}{\partial \rho}\right|_0 = 0, \quad \left.\frac{\partial \tilde{m}_{t2}^{(0)}}{\partial \rho}\right|_0 = 1, \tag{A2b}$$

$$\tilde{m}_{\rho 3}^{(0)}(R) = \tilde{m}_{t3}^{(0)}(R) = 0, \quad \tilde{m}_{\rho 4}^{(0)}(R) = 0, \tilde{m}_{t4}^{(0)}(R) = 1, \tag{A2c}$$

$$\left.\frac{\partial \tilde{m}_{\rho 3}^{(0)}}{\partial \rho}\right|_R = 1, \quad \left.\frac{\partial \tilde{m}_{t3}^{(0)}}{\partial \rho}\right|_R = 0, \quad \left.\frac{\partial \tilde{m}_{\rho 4}}{\partial \rho}\right|_R = \left.\frac{\partial \tilde{m}_{t4}}{\partial \rho}\right|_R = 0. \tag{A2d}$$

Such solutions of the homogeneous equation exist for any value of the frequency

The Green's functions we seek may then be written in the form

$$G_{\rho,t}^{(0)}(\rho,\rho') = g_{\rho,t}^{<}(\rho,\rho')\,\theta(\rho'-\rho) + g_{\rho,t}^{>}(\rho,\rho')\,\theta(\rho-\rho'), \tag{A3}$$

where $\theta(x)$ is the Heaviside step function. We assert that the functions $g^{>}(\rho,\rho')$ and $g^{<}(\rho,\rho')$ can be written in the form

$$g_{\rho,t}^{<}(\rho,\rho') = \tilde{m}_{\rho,t1}^{(0)}(\rho)a_1(\rho') + \tilde{m}_{\rho,t2}^{(0)}(\rho)a_2(\rho') \tag{A4a}$$

and

$$g_{\rho,t}^{>}(\rho,\rho') = \tilde{m}_{\rho,t3}^{(0)}(\rho)a_3(\rho') + \tilde{m}_{\rho,t4}^{(0)}(\rho)a_4(\rho'). \tag{A4b}$$

If these forms are inserted into Eq. (9) and the conditions in Eq. (A2) are noted then the boundary conditions for $m_{\rho,t}^{(0)}(\rho)$ stated in Sec. II are automatically obeyed. We next turn to an argument from which the four functions $a_i(\rho')$ may be determined.

We may determine the functions $a_{\rho,ti}(\rho')$ by matching $g_{\rho,t}^{>}(\rho,\rho')$ to $g_{\rho,t}^{<}(\rho,\rho')$ at the point $\rho=\rho'$ and by also utilizing the jump condition on the derivatives, which reads $\partial g_t^{>}(\rho,\rho')/\partial\rho|_{\rho'} - \partial g_t^{<}(\rho,\rho')/\partial\rho|_{\rho'} = +1/D$ while $\partial g_\rho^{>}(\rho,\rho')/\partial\rho|_{\rho'} - \partial g_\rho^{<}(\rho,\rho')/\partial\rho|_{\rho'} = 0$. These conditions lead to four inhomogeneous linear equations which—when inverted—allow the functions $a_i(\rho')$ to be expressed in terms of the known functions $\tilde{m}_{\rho,ti}^{(0)}(\rho')$. These equations may be written in the form

$$M(\rho') \cdot A(\rho') = B, \tag{A5}$$

where $M(\rho')$ is a 4×4 matrix whose elements are the functions $\tilde{m}_{\rho,ti}^{(0)}(\rho')$ and the derivatives $\partial\tilde{m}_{\rho,ti}^{(0)}/\partial\rho|_{\rho'}$,

$$M(\rho') = \begin{pmatrix} \tilde{m}_{t1}^{(0)}(\rho'), \tilde{m}_{t2}^{(0)}(\rho'), -\tilde{m}_{t3}^{(0)}(\rho'), -\tilde{m}_{t4}^{(0)}(\rho') \\ \tilde{m}_{\rho 1}^{(0)}(\rho'), \tilde{m}_{\rho 2}^{(0)}(\rho'), -\tilde{m}_{\rho 3}^{(0)}(\rho'), -\tilde{m}_{\rho 4}^{(0)}(\rho') \\ \tilde{m}_{t1}'^{(0)}(\rho'), \tilde{m}_{t2}'^{(0)}(\rho'), -\tilde{m}_{t3}'^{(0)}(\rho'), -\tilde{m}_{t4}'^{(0)}(\rho') \\ \tilde{m}_{\rho 1}'^{(0)}(\rho'), \tilde{m}_{\rho 2}'^{(0)}(\rho'), -\tilde{m}_{\rho 3}'^{(0)}(\rho'), -\tilde{m}_{\rho 4}'^{(0)}(\rho') \end{pmatrix}, \tag{A6}$$

and

$$A(\rho') = \begin{pmatrix} a_1(\rho') \\ a_2(\rho') \\ a_3(\rho') \\ a_4(\rho') \end{pmatrix}$$

with

$$B = \begin{pmatrix} 0 \\ 0 \\ -1/D \\ 0 \end{pmatrix}.$$

The functions $a_i(\rho')$ are found by inverting the matrix in Eq. (A5). When this is done, of course one encounters the determinant of the matrix $M(\rho')$, $W(\rho')=\mathrm{Det}[M(\rho')]$, which we call the Wronskian since it is a generalization of the Wronskian encountered in classical Sturm-Liouville theory. We refer to this quantity as the Wronskian in the text, and in what follows here. We conclude with the statement of a useful theorem which serves as a generalization of the standard textbook theorem regarding the behavior of the Wronskian encountered in the construction of the Green's function in standard Sturm-Liouville theory. Suppose we have a set of N functions $\{F_n(\rho)\}$ which satisfy differential equations of the form

$$\frac{dF_n(\rho)}{d\rho} = \sum_{l=1}^{N} p_{nl}(\rho)F_l(\rho). \tag{A7}$$

Equation (A7) will admit N sets of linearly independent solutions and the jth set will be labeled $\{F_n^{(j)}(\rho)\}$. Let $Q(\rho)$ be the matrix

$$Q(\rho) = \begin{pmatrix} F_1^{(1)}(\rho)F_1^{(2)}(\rho)\cdots F_1^{(N)}(\rho) \\ F_2^{(1)}(\rho)F_2^{(2)}(\rho)\cdots F_2^{(N)}(\rho) \\ \vdots \\ \vdots \\ \vdots \\ F_N^{(1)}(\rho)F_N^{(2)}(\rho)\cdots F_N^{(N)}(\rho) \end{pmatrix}. \tag{A8}$$

We then have the following theorem:

$$\frac{d}{d\rho}\mathrm{Det}\ Q(\rho) = \mathrm{Tr}\ P(\rho)\mathrm{Det}\ Q(\rho), \tag{A9}$$

where $\mathrm{Tr}\ P(\rho)=\Sigma_{l=1}^{N} p_{ll}(\rho)$.

Now, we may apply this theorem by making the identification $F_1^{(i)}(\rho)=\tilde{m}_{\rho i}^{(0)}(\rho), F_2^{(i)}(\rho)=\tilde{m}_{ti}^{(0)}(\rho), F_3^{(i)}(\rho)=\partial\tilde{m}_{\rho i}^{(0)}/\partial\rho$, $F_4^{(i)}(\rho)=\partial\tilde{m}_{ti}^{(0)}/\partial\rho$ (the negative signs on the third and fourth columns of Eq. (A6) are innocuous as far as the determinant is concerned). When Eq. (A1) is written in terms of the set $\{F_n^{(i)}(\rho)\}$ it has a form compatible with those in Eq. (A7), and the matrix $M(\rho)$ has the form of $Q(\rho)$ in Eq. (A8). One finds that $\mathrm{Tr}\ P(\rho)=-2/\rho$ so that $\mathrm{Det}[M(\rho)]=W(\rho)=CR^2/\rho^2$. The constant $C=C(\Omega)$ may be evaluated by computing $W(R)$ so we have

$$W(\rho) = \left(\frac{R}{\rho}\right)^2 W(R). \tag{A10}$$

The structure of $C(\Omega)=W(R)$ as a function of frequency is used in this paper in order to obtain the frequencies and linewidths of the modes.

The proof of the theorem in Eq. (A9) is sketched in Appendix B.

APPENDIX B: PROOF OF EQ. (A9)

In what follows we use the summation convention wherein one sums over repeated indices. For simplicity, we also confine our attention to the case where there are four functions in our set $\{F_n^{(j)}\}$. The extension of the proof to the case where we have N functions in the set is straightforward. Thus, Eq. (A7) is written

$$\frac{\partial F_i^{(j)}}{\partial\rho} = p_{il}F_l^{(j)}. \tag{B1}$$

Then after we form the matrix $Q(\rho)$ defined in Appendix A, we have

$$\mathrm{Det}\ Q = \varepsilon_{ijkn}F_i^{(1)}F_j^{(2)}F_k^{(3)}F_n^{(4)}. \tag{B2}$$

where ε_{ijkn} is the Levi-Civita tensor of rank four, which vanishes when any two indices are equal, and which equals +1 or −1 depending on whether $ijkn$ is an even or odd permutation of 1234. Clearly, if $D=\mathrm{Det}(Q)$,

$$\frac{dD}{d\rho} = \varepsilon_{ijkn}p_{il}F_l^{(1)}F_j^{(2)}F_k^{(3)}F_n^{(4)} + \varepsilon_{ijkn}p_{jl}F_i^{(1)}F_l^{(2)}F_k^{(3)}F_n^{(4)} + \ldots \tag{B3}$$

In the terms which involve p_{il} with $l\neq i$, l has to be equal to j, k, or n and of course i, j, k, and n must all be different by virtue of ε_{ijkn}. A similar statement applies to the term which involves p_{jl}. Suppose in the first term on the right-hand side of Eq. (B3), we consider the term with $l=j$, which has the form $\varepsilon_{ijkn}p_{ij}F_j^{(1)}F_j^{(2)}F_k^{(3)}F_n^{(4)}$. We compare this with the contribution from the second term for which $l=i$. This has the form $\varepsilon_{ijkn}p_{ji}F_i^{(1)}F_i^{(2)}F_k^{(3)}F_n^{(4)}$. Upon interchanging the summation indices i and j in the last-mentioned term, it becomes $\varepsilon_{jikn}p_{ij}F_j^{(1)}F_j^{(2)}F_k^{(3)}F_n^{(4)}$, and we see that it exactly cancels the first-mentioned term. Similarly, all other couples of this type cancel each other.

Thus the only terms in Eq. (B3) which survive are the terms which come from the diagonal elements of p_{il}. Thus, Eq. (B3) becomes

$$\begin{aligned}\frac{dD}{d\rho} &= \varepsilon_{ijkn}p_{ii}F_i^{(1)}F_j^{(2)}F_k^{(3)}F_n^{(4)} + \varepsilon_{ijkn}p_{jj}F_i^{(1)}F_j^{(2)}F_k^{(3)}F_n^{(4)} + \ldots \\ &= (p_{ii}+p_{jj}+p_{kk}+p_{nn})\varepsilon_{ijkn}F_i^{(1)}F_j^{(2)}F_k^{(3)}F_n^{(4)} = \mathrm{Tr}(P)D,\end{aligned} \tag{B4}$$

where as in Appendix A, $\mathrm{Tr}(P)=\Sigma_{i=1}^{4}p_{ii}$.

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